2009
Leap Frog Relay Grades 9-12
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. The sum of the (positive integer) divisors of 2009 is ________.

   (a) 2391 (b) 2392

   (c) 2393 (d) 2394

   (e) None of these

   Solution. (d) The prime factorization of 2009 is $2009 = 7^2 \cdot 41$. Thus, the sum of the divisors of 2009 is

$$\begin{aligned}(1+7+7^2)(1+41) &= 57 \cdot 42 \\ &= 2394.\end{aligned}$$

2. The solution x to the equation $\log_2 \log_3 \log_5 x = 1$ satisfies ________.

   (a) $0 < x \leq 10$ (b) $10 < x \leq 100$

   (c) $100 < x \leq 1000$ (d) $1000 < x \leq 10000$

   (e) None of these

   Solution. (e) The solution to $\log_2 y = 1$ is $y = 2$. Hence, $2 = \log_3 \log_5 x$. The solution to $2 = \log_3 w$ is $w = 9$, hence $\log_5 x = 9$. The solution to this last equation is $x = 5^9$. Now, $5^9 = (5^4)^2 \cdot 5 = (625)^2 \cdot 5 > 100^2 = 10000$. Thus $x > 10000$, none of the answer choices provided.

3. The number of real distinct solutions to the equation

$$\frac{1}{x} - \frac{5}{x^2} + \frac{6}{x^3} = x^3 - 5x^2 + 6x$$

is ________.

(a) 1 (b) 2

(c) 3 (d) 4

(e) None of these

Solution. (d) Multiply each side of the equation by x^3, and rearrange the right hand side, to obtain the equation

$$x^2 - 5x + 6 = x^4(x^2 - 5x + 6)$$

Since $x^2 - 5x + 6 = (x-2)(x-3)$, we get the two real solutions $x = 2, 3$. Now divide the equation by $x^2 - 5x + 6$ to obtain $1 = x^4$, with the two additional solutions $x = -1, 1$. This gives us a total of 4 real solutions ($x = -1, 1, 2, 3$).

4. The base 2 representation of the number N is $(11 \cdots 11)_2$ (that is 2009 ones). What is the base 4 representation of N?

(a) $(33 \cdots 33)_4$ (1005 threes) (b) $(33 \cdots 33)_4$ (1004 threes)

(c) $(133 \cdots 33)_4$ (1005 threes) (d) $(133 \cdots 33)_4$ (1004 threes)

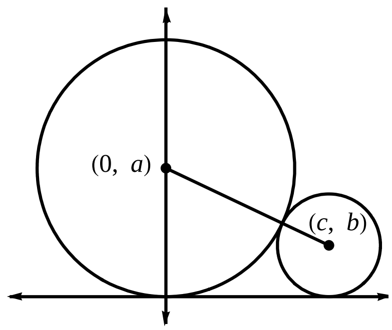
(e) None of these

Solution. (d)

$$\begin{aligned}
(11 \cdots 11)_2 &= 2^{2008} + 2^{2007} + 2^{2006} + \cdots + 2^1 + 2^0 \\
&= 2^{2008} + (2 \cdot 2^{2006} + 2^{2006}) + (2 \cdot 2^{2004} + 2^{2004}) \cdots + (2+1) \\
&= 4^{1004} + (2+1) \cdot 4^{1003} + (2+1) \cdot 4^{1002} + \cdots + (2+1) \cdot 4^0 \\
&= 4^{1004} + 3 \cdot 4^{1003} + 3 \cdot 4^{1002} + \cdots + 3 \cdot 4^0 \\
&= (133 \cdots 33)_4,
\end{aligned}$$

where there are 1004 threes.

5. In the figure below, the two circles are tangent and the respective centers $(0, a)$ and (c, b)) are connected by a line segment that goes through the point of tangency. The numbers a, b and c then satisfy the algebraic equation ________.



(a) $a^2 = 4bc$

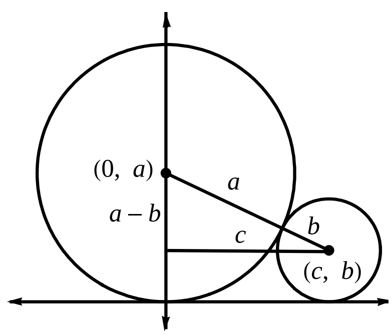
(b) $c^2 = 4ab$

(c) $a^2 + b^2 = c^2$

(d) $b^2 + c^2 = a^2$

(e) None of these

Solution. **(b)** Use the Pythagorean Theorem on the right triangle pictured.



$$(a-b)^2 + c^2 = (a+b)^2 \implies -2ab + c^2 = 2ab$$
$$\implies c^2 = 4ab.$$

6. Lenny sells the *ispud*©. Last week he sold 1000 *ispud*©s. Lenny wants to increase the price of an *ispud*© by 20%. What is the least number

he needs to sell this week at the increased price to make at least as much income as he made last week?

(a) 834

(b) 835

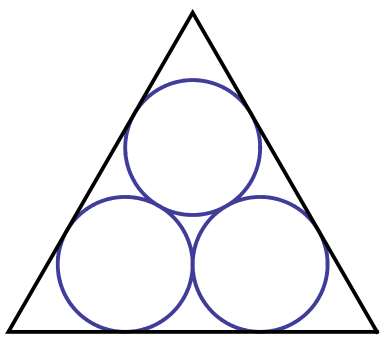
(c) 836

(d) 837

(e) None of these

Solution. (a) Let C be the cost of the *ispud*© last week. Then Lenny's income from last week's sales is $1000C$. If Lenny sells N *ispud*©s this week at the price $(1.2)C$, then to obtain a comparable income, it must be the case that $N(1.2)C \geq 1000C$. Thus, $N \geq 1000/1.2 = 833\frac{1}{3}$. Thus, the least number Lenny must sell is 834.

7. The equilateral triangle is circumscribing (tangent to) the three kissing (tangent) unit radius (radius = 1) circles. The area of the triangle is ________.



(a) $4 + 6\sqrt{3}$

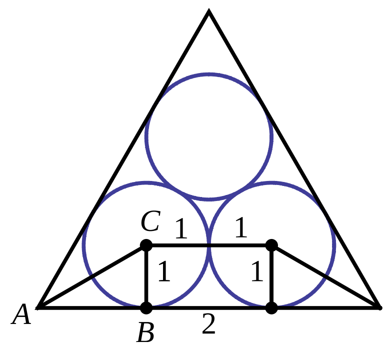
(b) $6 + 4\sqrt{3}$

(c) $3 + 5\sqrt{3}$

(d) $5 + 3\sqrt{3}$

(e) None of these

Solution. (b) The area enclosed by an equilateral triangle of side length s is $(\sqrt{3}/4)s^2$. So we need only determine the side length of the triangle. Note that $\triangle ABC$ in the figure below is a 30°-60°-90° triangle, and since $BC = 1$, it follows that $AB = \sqrt{3}$.



Thus the side length of the circumscribing triangle is $s = 2+2\sqrt{3}$. The area is then

$$\begin{aligned}
\text{Area} &= \frac{\sqrt{3}}{4}s^2 \\
&= \frac{\sqrt{3}}{4}(2+2\sqrt{3})^2 \\
&= \sqrt{3}(1+\sqrt{3})^2 \\
&= \sqrt{3}(4+2\sqrt{3}) \\
&= 6+4\sqrt{3}.
\end{aligned}$$

8. Let $f(x) = x^{2009}+x^{1776}$. Then $f(-1) = 0$, and hence $f(x) = (x+1)q(x)$ for some polynomial $q(x)$. Determine $q(-1)$.

(a) 0

(b) 233

(c) 1776

(d) 1

(e) None of these

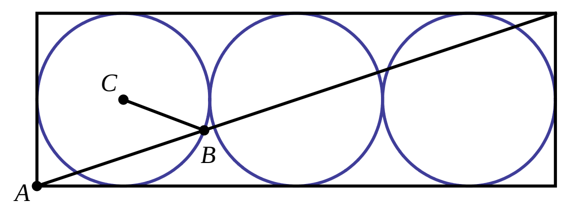
Solution. (b) We may write

$$\begin{aligned}
q(x) &= \frac{f(x)}{x+1} \\
&= \frac{x^{2009}+x^{1776}}{x+1} \\
&= \frac{x^{1776}(x^{233}+1)}{x+1}
\end{aligned}$$

$$= \quad x^{1776}(x^{232} - x^{231} + x^{230} - x^{229} + \cdots - x^1 + 1),$$

where, in the second factor, the coefficient of x^n is 1 if n is even and -1 if n is odd. It follows that $q(-1) = (-1)^{1776}(\underbrace{1+1+\cdots+1+1}_{233}) = 233$.

9. The three mutually tangent circles, all the same radius, are circumscribed by the rectangle. The diagonal of the rectangle intersects the left-most circle at the indicated point B. Let C be the center of the left-most circle and let A be the lower left-hand corner of the rectangle. Then $\sin \angle ABC =$ ________.



(a) $\frac{\sqrt{10}}{5}$

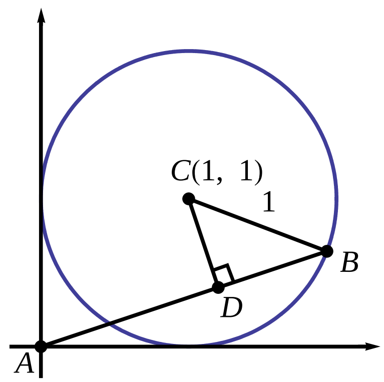
(b) $\frac{3}{5}$

(c) $\frac{2}{3}$

(d) $\frac{4}{5}$

(e) None of these

Solution. (a) We may assume the circles all have unit radius. Place the figure in a rectangular coordinate system where $A = (0, 0)$. Drop a perpendicular from C to a point $D \in \overline{AB}$. Then we have the following picture.



Then $BC = 1$ and so $\sin \angle ABC = CD$. The equation of the line $\overline{AB}$ is $y = x/3$ and so the equation of the line $\overline{CD}$ is $y - 1 = -3(x-1)$, which is equivalent to $y = -3x + 4$. Thus we can determine the coordinates of D by solving the pair of equations ($\overline{AB}$ and $\overline{CD}$) to get

$$D = \left(\frac{6}{5}, \frac{2}{5}\right).$$

Now we just need the distance formula to finish the problem.

$$\begin{aligned}
\sin \angle ABC &= CD \\
&= \sqrt{\left(\frac{6}{5} - 1\right)^2 + \left(\frac{2}{5} - 1\right)^2} \\
&= \sqrt{\frac{1}{25} + \frac{9}{25}} \\
&= \frac{\sqrt{10}}{5}.
\end{aligned}$$

10. There are 10 students in a room. There are 4 Freshman, 1 Sophomore, 3 Juniors, and 2 Seniors. Suppose 5 of the students are chosen at random. What is the probability that, among the chosen 5, the majority is Freshman?

(a) $\frac{11}{42}$ (b) $\frac{13}{42}$

(c) $\frac{15}{42}$ (d) $\frac{17}{42}$

(e) None of these

Solution. (a) The question is asking for the probability that of the 5 students chosen, 3 or 4 are Freshman. The number of ways to select 5 students from 10, is

$$\begin{aligned}\binom{10}{5} &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 252.\end{aligned}$$

The number of ways that you can choose 3 Freshman and 2 non-Freshman is

$$\begin{aligned}\binom{4}{3}\binom{6}{2} &= \left(\frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1}\right)\left(\frac{6 \cdot 5}{2 \cdot 1}\right) \\ &= 60.\end{aligned}$$

And the number of ways that you can choose 4 Freshman and 1 non-Freshman is

$$\begin{aligned}\binom{4}{4}\binom{6}{1} &= \left(\frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}\right)\left(\frac{6}{1}\right) \\ &= 6.\end{aligned}$$

Thus, the probability of choosing mostly Freshman is

$$\begin{aligned}\frac{60 + 6}{252} &= \frac{66}{252} \\ &= \frac{11}{42}\end{aligned}$$