2009
Leap Frog Relay Grades 9-12
Part II Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

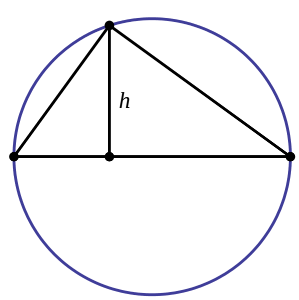
1. What is the largest area enclosed by a triangle that is inscribed in a unit circle (radius = 1) in such a way that one of the triangle sides is a diameter of the circle?

   (a) $\frac{2}{3}$ (b) $\frac{3}{4}$

   (c) $\frac{4}{5}$ (d) 1

   (e) None of these

   Solution. **(d)** The area of the triangle will be $\frac{1}{2} \cdot 2 \cdot h = h$, where h is the height of the triangle (see figure below).

   So the largest area will occur when $h = 1$, the radius of the circle. Hence the largest area will be equal to 1.

2. Given the base-b arithmetic identity $4214_b - 3141_b = 1043_b$, the value of the base b is ________.

(a) 8 (b) 7

(c) 6 (d) 5

(e) None of these

Solution. (b) By definition, $4214_b = 4b^3 + 2b^2 + b + 4$, $3141_b = 3b^3 + b^2 + 4b + 1$ and $1043_b = b^3 + 4b + 3$. So the identity $4214_b - 3141_b = 1043_b$ translates to $(4b^3 + 2b^2 + b + 4) - (3b^3 + b^2 + 4b + 1) = b^3 + 4b + 3$. Simplifying this equality gives $b^2 - 7b = 0$. The non-zero solution to this equation is $b = 7$.

3. A runner (R) and a walker (W) are positioned on diametrically opposite positions on a circular track. On the track, R is twice as fast as W. Assume they start simultaneously, move at constant velocity on the circular track, never change directions while R runs counterclockwise and W walks clockwise. How many times do R and W meet on the track in the time it takes W to go around the track two times?

(a) 4 (b) 6

(c) 8 (d) 10

(e) None of these

Solution. (b) Superimpose a 12-hour clock on the track. R starts at position 12 and W starts at position 6. Since R moves twice as fast as W, they meet in following positions (in order): 8, 12, 4, 8, 12, 4–which is 6 times.

4. Let T be the area of an isosceles right triangle and let C be the area of the corresponding inscribed circle to this triangle. Then, $C/T =$ ________.

(a) $(3-2\sqrt{2})\pi$

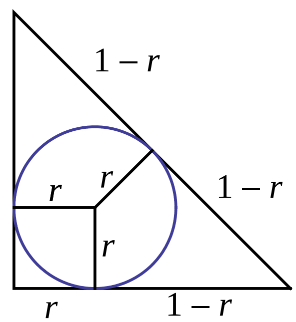
(b) $\dfrac{(4-2\sqrt{2})\pi}{5}$

(c) $\dfrac{(5-2\sqrt{2})\pi}{8}$

(d) $\dfrac{(5-2\sqrt{2})\pi}{10}$

(e) None of these

Solution. **(a)** Since we are computing the ratio of areas, we may assume the two leg lengths of the isosceles right triangle are both equal to 1. Let r be the radius of the inscribed circle. The situation, with the indicated lengths, is pictured below.



Since the hypotenuse of the right triangle is $\sqrt{2}$, we must have $2-2r = \sqrt{2}$. Solving for r gives

$$r = 1 - \frac{1}{\sqrt{2}}.$$

Thus,

$$\begin{aligned} \frac{C}{T} &= \frac{\pi r^2}{\frac{1}{2}\cdot 1 \cdot 1} \\ &= 2\left(1-\frac{1}{\sqrt{2}}\right)^2 \pi \\ &= (3-2\sqrt{2})\pi. \end{aligned}$$

5. Assume the two parabolas $y = a_1x^2 + b_1x + c_1$, and $y = a_2x^2 + b_2x + c_2$ (with $a_1 \neq$ and $a_2 \neq 0$) share the same vertex. Then, it is *necessary* that…

(a) $a_1 + b_1 = a_2 + b_2$

(b) $a_1 + b_2 = a_2 + b_1$

(c) $a_1 b_2 = a_2 b_1$

(d) $a_1 b_1 = a_2 b_2$

(e) None of these

Solution. (c) Completing the square, we get

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c. \end{aligned}$$

Thus, the vertex of the parabola $y = ax^2 + bx + c$ is

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right).$$

So, if $y = a_1x^2 + b_1x + c_1$, and $y = a_2x^2 + b_2x + c_2$ share the same vertex, then

$$-\frac{b_1}{2a_1} = -\frac{b_2}{2a_2} \qquad \text{and} \quad -\frac{b_1^2}{4a_1} + c_1 = -\frac{b_2^2}{4a_2} + c_2.$$

The first equality gives $a_1 b_2 = a_2 b_1$, which gives a necessary condition. The other three answer choices can be eliminated using the two parabolas $y = 4x^2 - 8x$ and $y = x^2 - 2x - 3$, which share the same vertex $(1, -4)$.

6. Simplify the product

$$(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8).$$

(a) $\log_{27} 33$

(b) 1

(c) $\log_2 163$

(d) 4

(e) None of these

Solution. **(e)** Convert all of the logarithms to base 2 using the change-of-base formula:

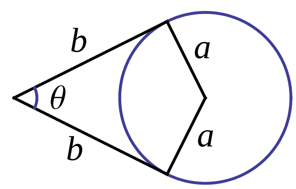
$$\log_b a = \frac{\log_2 a}{\log_2 b}.$$

Then,

$$\begin{aligned}(\log_2 3)(\log_3 4)(\log_4 5)\log_5 6)(\log_6 7)(\log_7 8) &= \log_2 3 \cdot \frac{\log_2 4}{\log_2 3} \cdot \frac{\log_2 5}{\log_2 4} \cdot \frac{\log_2 6}{\log_2 5} \cdot \frac{\log_2 7}{\log_2 6} \cdot \frac{\log_2 8}{\log_2 7} \\ &= \log_2 8 \\ &= 3,\end{aligned}$$

none of the answer choices provided.

7. In the picture below, the two line segments of length b are each tangent to the circle of radius a and meet in an angle θ. Then, $\sin\theta =$ ________.

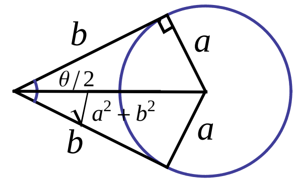


(a) $\dfrac{ab}{a^2+b^2}$ (b) $\dfrac{ab}{\sqrt{a^2+b^2}}$

(c) $\dfrac{1}{\sqrt{a^2+b^2}}$ (d) $\dfrac{1}{a^2+b^2}$

(e) None of these

Solution. **(e)** The segment that joins the intersection of the two tangent segments and the center of the circle bisects the angle θ. Also, the tangent is perpendicular to the radius, forming the right triangle pictured below.



Finally, use the double angle formula for sine,

$$\begin{aligned}\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ &= 2\cdot\frac{a}{\sqrt{a^2+b^2}}\cdot\frac{b}{\sqrt{a^2+b^2}}\\ &= \frac{2ab}{a^2+b^2},\end{aligned}$$

none of the answer choices provided.

8. Upon dividing 99^{2009} by 100, the remainder is ______.

(a) 1

(b) 99

(c) 9

(d) 19

(e) None of these

Solution. **(b)** Use the binomial expansion,

$$\begin{aligned}99^{2009} &= (100-1)^{2009}\\ &= \sum_{n=0}^{2009}\binom{2009}{n}100^{2009-n}(-1)^n.\end{aligned}$$

Note that every term in the sum is a multiple of 100 except the last term, which is -1. So, upon dividing by 100, the remainder will be $100-1=99$.

9. If it takes x painters to paint a wall in y days (all at the same rate), how many days does it take z painters to paint the same wall?

(a) $\dfrac{xy}{z^2}$

(b) $\dfrac{yz}{x^2}$

(c) $\dfrac{xz}{y}$

(d) $\dfrac{xy}{z}$

(e) None of these

Solution. **(d)** One painter can paint $\frac{1}{x}$ of the wall in y days. And so, one painter can paint $\frac{1}{xy}$ of the wall in one day. Consequently, z

painters can paint $\frac{z}{xy}$ of the wall in one day. So it would take z painters $\frac{xy}{z}$ days to paint the wall.

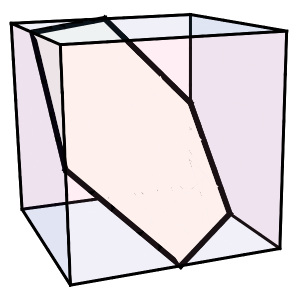
10. A plane intersects a $1' \times 1' \times 1'$ cube in a regular hexagon. What is the area of the hexagon? (A regular hexagon is a six sided polygon, all of whose side lengths are equal in measure.)

(a) $\frac{3\sqrt{3}}{4}$ ft^2 (b) $\frac{3\sqrt{2}}{4}$ ft^2

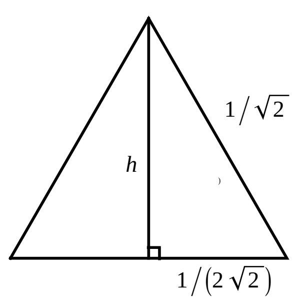
(c) $\sqrt{3}$ ft^2 (d) $\sqrt{2}$ ft^2

(e) None of these

Solution. **(a)** In order for the intersection to be a regular hexagon, the plane must intersect six of the edges at midpoints.



By the Pythagorean Theorem, each side length of the hexagon is $\frac{1}{\sqrt{2}}$ feet. To compute the area of the hexagon, divide it into six congruent equilateral triangles.



Again, by the Pythagorean Theorem, the altitude of this triangle is

$$h = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{2\sqrt{2}}\right)^2} = \frac{\sqrt{3}}{2\sqrt{2}}.$$

Hence, the area of the triangle is (in square feet)

$$\text{Triangle Area} = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}}{8}.$$

So the hexagon area is (in square feet)

$$\text{Hexagon Area} = 6 \cdot \frac{\sqrt{3}}{8} = \frac{3\sqrt{3}}{4}.$$