2008
Leap Frog Relay Grades 9-12
Part I Solutions

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. $\log_9 \log_8 \log_5 25 =$ ______

(a) $\frac{1}{3}$ (b) $-\frac{2}{3}$

(c) $\frac{1}{2}$ (d) $-\frac{3}{2}$

(e) None of these

Solution. (e)

$$\begin{aligned} \log_9 \log_8 \log_5 25 &= \log_9 \log_8 2 \\ &= \log_9 \frac{1}{3} \\ &= -\frac{1}{2}, \end{aligned}$$

none of the answer choices provided.

2. The base-2 number (repeated decimal) $.\overline{01}_2 = .010101\ldots_2$ is equal to ______.

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$

(c) $\frac{1}{5}$ (d) $\frac{1}{6}$

(e) None of these

Solution. (a)

$$.010101\ldots_2 = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \\
&= \frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right) \\
&= \frac{1}{3}.
\end{aligned}
$$

3. Lenny has a jar of coins. The 1000 coins in the jar consists of nickels, dimes and quarters (at least one of each). Assuming that the value of the coins is exactly 100 dollars, what is the ratio of the number of nickels to the number of quarters?

   (a) 3
   (b) 4
   (c) 5
   (d) 6
   (e) None of these

   Solution. (a) Let N, D and Q be the respective numbers of nickels, dimes and quarters. There are 1000 coins whose total value is \$100, which gives the two equations,

$$
\begin{aligned}
N + D + Q &= 1000 \\
5N + 10D + 25Q &= 10000.
\end{aligned}
$$

   Divide the second equation by 5,

$$N + 2D + 5Q = 2000,$$

   and multiply the first equation by 2,

$$2N + 2D + 2Q = 2000.$$

   Thus the left hand sides of the above two equations are equal,

$$N + 2D + 5Q = 2N + 2D + 2Q.$$

   This simplifies to

$$3Q = N,$$

   so the ratio of nickels to quarters is 3.

4. Suppose $\sin(\theta + 90^\circ) = \cos(\theta + 45^\circ)$. Then $\tan\theta =$ ________.

   (a) $2 - \sqrt{2}$
   (b) $\sqrt{2} - 2$
   (c) $1 - \sqrt{2}$
   (d) $\sqrt{2} - 1$
   (e) None of these

Solution. **(c)** Using the angle addition formulas, we get

$$\begin{aligned} \sin(\theta + 90^\circ) &= \cos\theta \quad \text{and} \\ \cos(\theta + 45^\circ) &= \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta) \end{aligned}$$

So we have the equation

$$\cos\theta = \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)$$

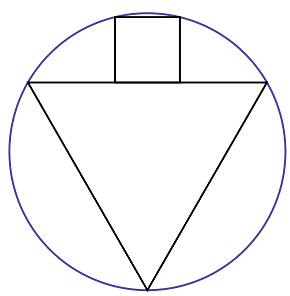
which can be rewritten as

$$(1 - \sqrt{2})\cos\theta = \sin\theta.$$

We know that $\cos\theta \neq 0$ for otherwise the above equation would imply $\sin\theta = 0 = \cos\theta$, which is impossible. So we may divide by $\cos\theta$ to obtain

$$\tan\theta = 1 - \sqrt{2}.$$

5. In the accompanying figure, the circle has radius equal to 1 unit, the inscribed triangle is equilateral and the inscribed rectangle is a square. The side length of the square is then ______ units.



(a) $\frac{1}{\sqrt{2}}$

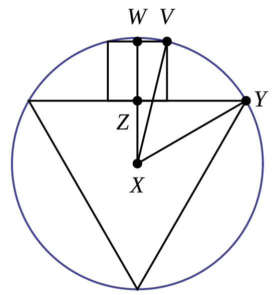
(b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$

(d) $\frac{1}{\sqrt{3}}$

(e) None of these

Solution. **(e)** Label the figure as pictured, where X denotes the center of the circle and the segment $\overline{WX}$ is perpendicular to the segment $\overline{WV}$.



By assumption, $XY = 1$. Since the triangle is equilateral, $\triangle XYZ$ is a 30–60–90 triangle, and consequently $XZ = 1/2$. Let s denote the side length of the square, $s = WZ$. Then $WX = 1/2 + s$. On the other hand, WX is one leg of the right triangle XVW, whose hypotenuse is $XV = 1$ and whose other leg is $WV = s/2$. So by the pythagorean theorem, $WX = \sqrt{1 - s^2/4}$. Setting the two values for WX equal, we obtain an equation that we can solve for s,

$$\frac{1}{2} + s = \sqrt{1 - \frac{s^2}{4}}.$$

Square both sides of the equation, rearrange terms and multiply by 4 to get

$$5s^2 + 4s - 3 = 0.$$

By the pythagorean theorem,

$$\begin{aligned} s &= \frac{-4 \pm \sqrt{4^2 - 4(5)(-3)}}{2(5)} \\ &= \frac{-2 \pm \sqrt{19}}{5}. \end{aligned}$$

Choose the + square root to obtain a positive answer,

$$s = \frac{-2 + \sqrt{19}}{5},$$

none of the answer choices provided.

6. $\sqrt{5 - 2\sqrt{6}} =$ ________.

(a) $4 - 2\sqrt{3}$

(b) $3 - 2\sqrt{2}$

(c) $2 - \sqrt{3}$

(d) $\sqrt{3} - \sqrt{2}$

(e) None of these

Solution. (d)

$$\begin{aligned} 5 - 2\sqrt{6} &= 3 - 2\sqrt{6} + 2 \\ &= \left(\sqrt{3} - \sqrt{2}\right)^2. \end{aligned}$$

Take the positive square root to get

$$\sqrt{5 - 2\sqrt{6}} = \sqrt{3} - \sqrt{2}.$$

7. The average of five 100 point tests is 96. What is the smallest possible average of four of the tests?

(a) 95 (b) 90

(c) 85 (d) 80

(e) None of these

Solution. (a) The smallest average of four will occur if the fifth is as large as possible, namely 100. Since the average of the five is 96, the sum of the points of the five tests will be $5 \times 96 = 480$. If one score is 100, then the remaining four scores will sum to $480 - 100 = 380$, whose average is $380/4 = 95$.

8. You flip a fair coin successively until it comes up Heads. What is the probability that it takes at most 3 flips to come up Heads?

(a) $\frac{5}{8}$ (b) $\frac{7}{8}$

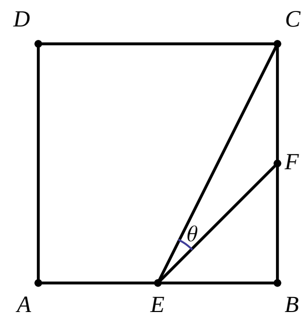
(c) $\frac{3}{4}$ (d) $\frac{5}{6}$

(e) None of these

Solution. (b) The answer is one minus the probability that the first 3 flips are Tails. The probability that the first three flips are tails is $\frac{1}{8}$. Thus the probability of no more than 3 flips is

$$1 - \frac{1}{8} = \frac{7}{8}.$$

9. Quadrilateral $ABCD$ is a square. The points E and F are the respective midpoints of $\overline{AB}$ and $\overline{BC}$. Suppose $\theta = \text{m}\angle FEC$. Determine $\tan\theta$.



(a) $\sqrt{\dfrac{2}{3}}$

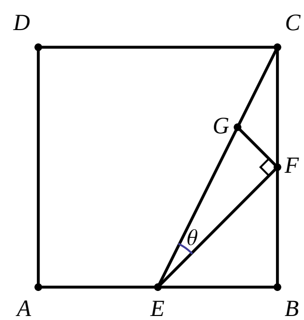
(b) $\dfrac{\sqrt{2}}{3}$

(c) $\dfrac{1}{3}$

(d) $\dfrac{1}{\sqrt{3}}$

(e) None of these

Solution. **(c)** Draw point G on $\overline{EC}$ so that $\overline{EF}$ is perpendicular to $\overline{GF}$.



Then $\tan\theta = GF/EF$. We may scale the square so that the side lengths are equal to 2 units. So $EB = BF = 1$, and we have, by the pythagorean theorem,

$$EF = \sqrt{2}.$$

So we are left with determining GF. Place the figure in a (x, y)–coordinate system so that E is the origin. Then $\overline{GF}$ has equation $y = -x + 2$ and $\overline{EC}$ has equation $y = 2x$. Solving these two equations gives the coordinates of $G = (2/3, 4/3)$. We can then use the distance formula to compute GF (where $F = (1, 1)$),

$$\begin{aligned} GF &= \sqrt{\left(\frac{2}{3} - 1\right)^2 + \left(\frac{4}{3} - 1\right)^2} \\ &= \frac{\sqrt{2}}{3}. \end{aligned}$$

Thus

$$\tan\theta = \frac{GF}{EF}$$

$$
\begin{aligned}
&= \frac{\sqrt{2}/3}{\sqrt{2}} \\
&= \frac{1}{3}.
\end{aligned}
$$

Alternative Solution. Let $\alpha = \angle BEC$ and $\beta = \angle BEF$. Then

$$
\begin{aligned}
\tan\theta &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \\
&= \frac{2-1}{1+2\cdot 1} \\
&= \frac{1}{3}.
\end{aligned}
$$

10. Let a, b and c be the three roots of $x^3 + 2x + 3 = 0$. Then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} = \underline{\qquad\qquad}.$$

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$

(c) $\frac{1}{3}$ (d) $\frac{1}{2}$

(e) None of these

Solution. (a) First note that

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ac)x - (abc),$$

Comparing this to $x^3 + 2x + 3$, we obtain the three equations

$$
\begin{aligned}
a+b+c &= 0 \\
ab+bc+ac &= 2 \\
abc &= -3.
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} &= -\left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right) \\
&= -\left(\frac{ab+bc+ac}{abc}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\left(\frac{2}{-3}\right) \\
&= \frac{2}{3}.
\end{aligned}
$$