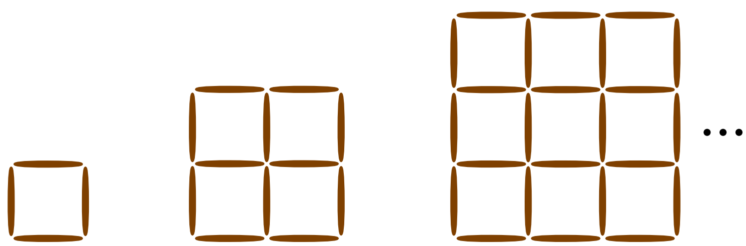
Leap Frog Relay I Solutions–2006

No calculators allowed
Correct Answer = 4, Incorrect Answer = −1, Blank = 0

1. Pictured below is the first three in a sequence of figures. Each is a checkerboard pattern of squares created using toothpicks. How many toothpicks are needed to create the tenth figure in the pattern?



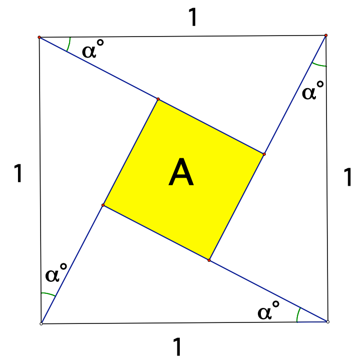
(a) 200 (b) 210

(c) 216 (d) 220

(e) None of these

Solution. (d) When counting the toothpicks divide the count into horizontal and vertical toothpicks. By symmetry there are the same number of horizontal as vertical. How many horizontal are there? There are 11 rows of 10 toothpicks each, for a total of 110. Thus there are $2 \cdot 110 = 220$ toothpicks in the tenth figure.

2. The area of the small square in the figure below is $A =$ ________ (square units).



(a) $\cos^2(2\alpha)$

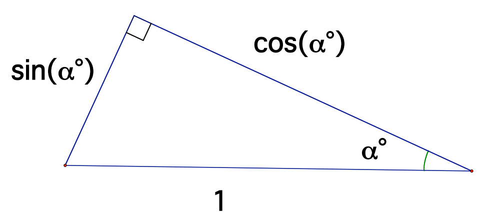
(b) $\cos(2\alpha)$

(c) $1 - \sin^2(2\alpha°)$

(d) $1 - \sin(2\alpha°)$

(e) None of these

Solution. (d) The region inside the large square but outside the small square is seen to be the sum of 4 mutually congruent right triangles. One of the triangles is pictured below.



The area of this triangle is $\frac{1}{2}\sin(\alpha°)\cos(\alpha°) = \frac{1}{4}\sin(2\alpha°)$ square units. Since the 4 triangles have equal area, the area of the small square is

$$1 - 4 \times \frac{1}{4}\sin(2\alpha°) = 1 - \sin(2\alpha°)$$

square units.

3. Tom Swift is running around a circular race track while John Walker is walking around the same track in the opposite direction. Assuming Tom can run three times as fast as John can walk, and that they start at the same time in the same place, how many times do they meet in the time it takes John to walk around the track 12 times? (Count their starting position and ending position as meetings.)

   (a) 36 (b) 37

   (c) 46 (d) 47

   (e) None of these

   Solution. (e) Each time John walks around the track he meets Tom on the start and end and 3 intermediate points. If we count the intermediate points and the end together, that's 4, not counting the start. So if John walks 1 time around, he meets Tom $1 + 4 = 5$ times. For 12 walks around, he meets Tom $1 + 12 \times 4 = 49$ times.

4. Suppose a and b are positive real numbers and R is the region in the cartesian plane defined by

$$\frac{|x|}{a} + \frac{|y|}{b} \leq 1.$$

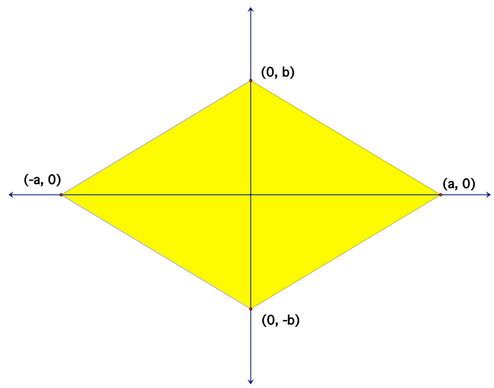
   The area enclosed by R is equal to _______ (square units).

   (a) $a^2 + b^2$ (b) $2ab$

   (c) $a + b$ (d) $2a^2b^2$

   (e) None of these

   Solution. (b) A sketch of R is pictured below.



The figure can easily be divided into four equal sized triangles with respective base length a and height b. So the numerical value of the area of the region is

$$4 \times \frac{ab}{2} = 2ab.$$

5. The units digit of the number $9^{2006} - 3^{2006}$ is ________.

(a) 6

(b) 4

(c) 2

(d) 0

(e) None of these

Solution. **(c)** First factor $9^{2006} - 3^{2006} = 3^{2006}(3^{2006} - 1)$. So what we need is the units digit of 3^{2006}. The units digit of powers of 3 follow a pattern that cycles every four powers: $3 \to 9 \to 7 \to 1 \to 3 \to \cdots$. Since the remainder upon division by 4 of 2006 is 2, the units digit of 3^{2006} is that same as that of 3^2, which is 9. It follows that the units digit of $3^{2006}(3^{2006} - 1)$ is the units digit of $9 \cdot 8 = 72$, which is 2.

6. Let x be a solution to the equation $\sin(x)\cos(2x) = -\sin(2x)\cos(x)$ for which $0 < x < \pi/2$ (in radians). Then $\tan(x) =$ ________.

(a) $\frac{\sqrt{3}}{2}$

(b) $\sqrt{3}$

(c) $2\sqrt{3}$

(d) $\frac{2}{\sqrt{3}}$

(e) None of these

Solution. **(b)** The double angle identities allow us to re-write the equation as

$$\sin(x)\left(\cos^2(x) - \sin^2(x)\right) = -2\sin(x)\cos(x)\cos(x).$$

Since $\sin(x) \neq 0$, we may divide out this term. A little more rearrangement gives

$$3\cos^2(x) = \sin^2(x).$$

We may divide by $\cos^2(x)$ (which also cannot be equal to zero) to get

$$\tan^2(x) = 3$$

and so $\tan(x) = \sqrt{3}$. (The negative square root can be ignored because $0 < x < \pi/2$.)

7. Let a_1 and a_2 be the two values of the real variable a for which the pair of equations $x + y = 4$, $(x - a)^2 + y^2 = 4$ has exactly one solution (x, y). The average $\frac{a_1 + a_2}{2}$ is equal to ________.

(a) 0

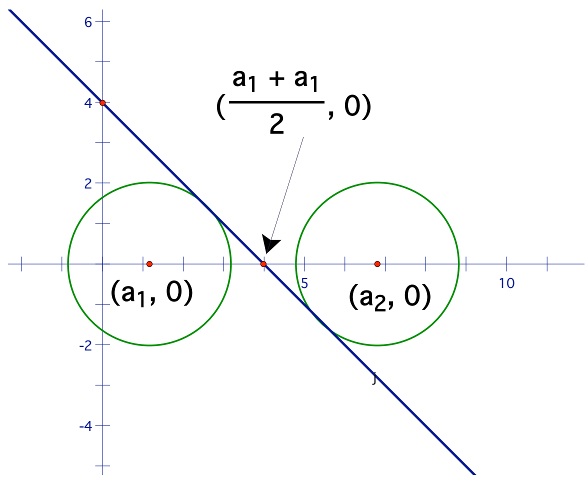
(b) $2\sqrt{2}$

(c) 4

(d) $\sqrt{2}$

(e) None of these

Solution. **(c)** The graph of $(x - a)^2 + y^2 = 4$ is a circle centered at $(a, 0)$ and with radius equal to 2. The two circles corresponding to the two values of a in the problem are pictured below, along with the line $x + y = 4$.



By the apparent symmetry in the figure, the average of a_1 and a_2 is equal to 4,

$$\frac{a_1 + a_2}{2} = 4.$$

8. If the natural numbers 0, 1, 2, ... are lined up to form a non-ending string of digits

$$01234567891011121314\ldots$$

then the 2006th digit in the string is ________.

(a) 9

(b) 0

(c) 3

(d) 7

(e) None of these

Solution. (d) The numbers 0 through 9 give 10 digits. The numbers 10 through 99 give 180 digits. So the numbers 0 through 99 contribute 190 digits. We need 1816 more digits. Since $1816 = 605 \times 3 + 1$, we know we can accomplish this with the 1st digit of the 606th number after 99. This number is 705. So the 2006th digit in the sequence is 7.

9. If $\mathrm{i} = \sqrt{-1}$, then $\left(\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}\right)^{2006} =$ ________

(a) $\frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}$

(b) $-\frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}$

(c) $-\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}$

(d) $\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}$

(e) None of these

Solution. **(c)** Since $\cos(60^\circ) = 1/2$ and $\sin(60^\circ) = \sqrt{3}/2$, we may write the problem as $(\cos(60^\circ) + \mathrm{i}\sin(60^\circ))^{2006}$. Now use De Moivre's rule,

$$\begin{aligned}
(\cos(60^\circ) + \mathrm{i}\sin(60^\circ))^{2006} &= \cos(2006 \times 60^\circ) + \mathrm{i}\ \sin(2006 \times 60^\circ) \\
&= \cos((334 \times 6 + 2) \times 60^\circ) + \mathrm{i}\sin((334 \times 6 + 2) \times 60^\circ) \\
&= \cos(334 \times 360^\circ + 120^\circ) + \mathrm{i}\sin(334 \times 360^\circ + 120^\circ) \\
&= \cos(120^\circ) + \mathrm{i}\sin(120^\circ) \\
&= -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}.
\end{aligned}$$

10. Suppose n, a and b are positive integers. In order for n to divide ab, it is ________ that n divides a or n divides b.

(a) necessary and sufficient

(b) necessary, but not sufficient

(c) sufficient, but not necessary

(d) neither necessary nor sufficient

(e) None of these

Solution. **(c)** It is not necessary because 4 divides $4 = 2 \times 2$, but 4 does not divide 2. It is sufficient however. This is because if say n divides say a, then $a = nm$ for some other positive integer m and hence $ab = nmb$ which is clearly divisible by n.