Problem of the Month
October 2009

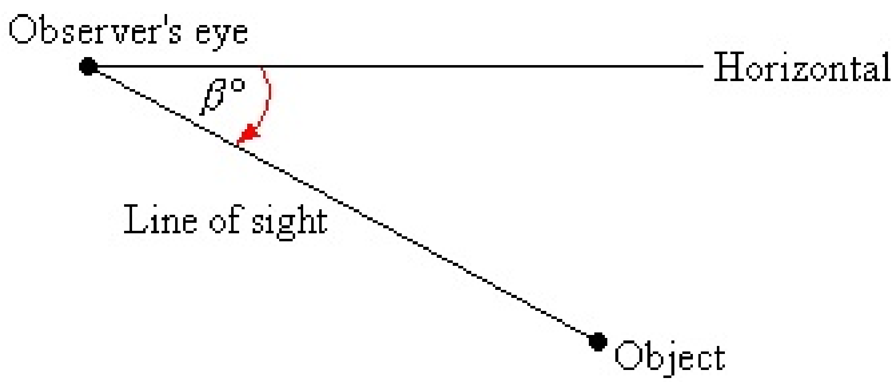
Problem for October 2009

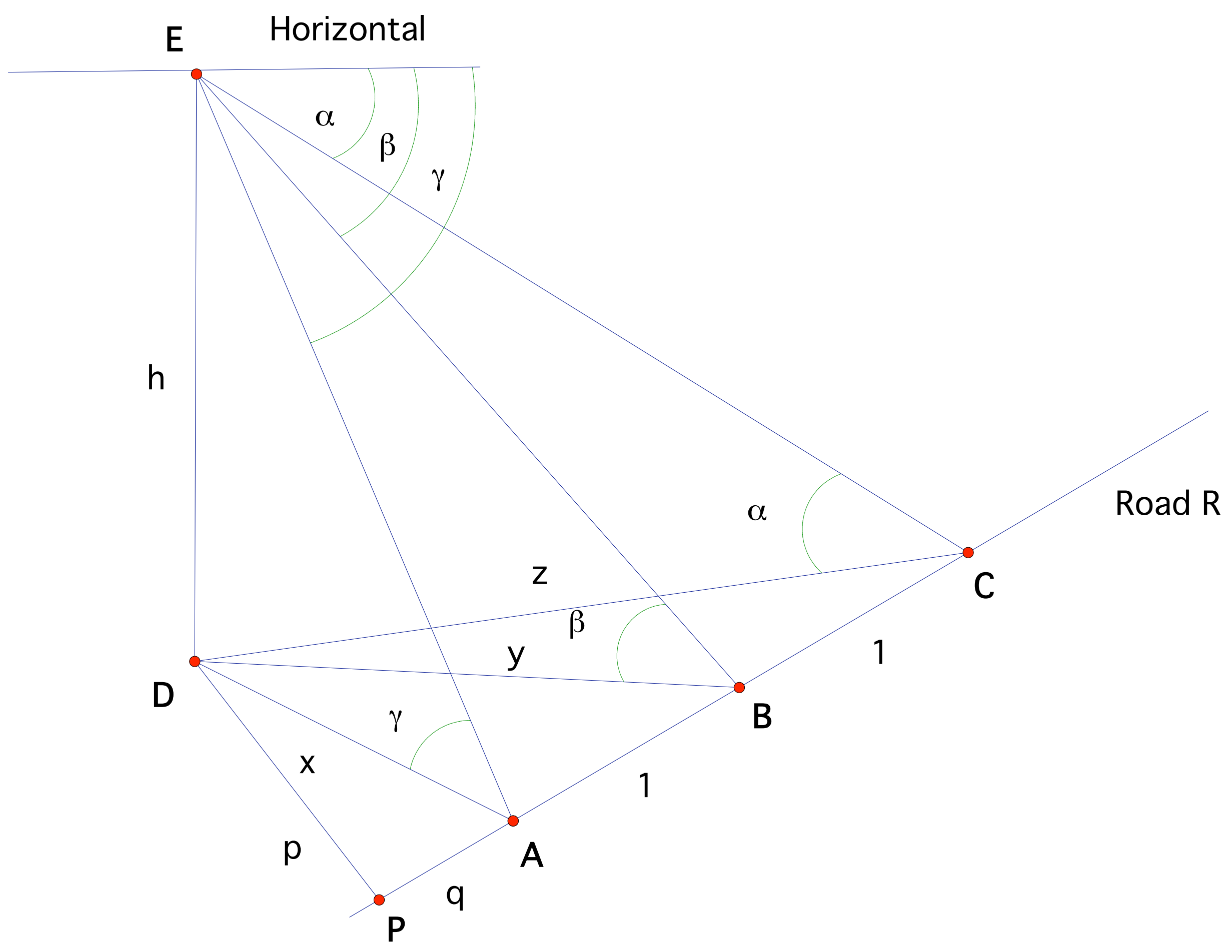
A person on the summit of a mountain observes that the angles of depression of a car moving on a straight road (which does not pass through the foot of the mountain) at three consecutive mile markers are α, β and γ respectively. Prove that the height of the mountain is

$$\left[\frac{2}{\cot^2\alpha - 2\cot^2\beta + \cot^2\gamma}\right]^{1/2}$$

<u>**Definition:**</u> The angle of depression is defined to be the angle made between the horizontal and the straight line of sight of an observer looking at an object that is at a lower height than the observer.

For example, β is the angle of depression in the picture below.





E
Horizontal
α
β
γ
h
α
Road R
z
C
β
y
1
D
B
γ
x
1
p
A
q
P

Solution:

Let DE represent the mountain with D as the foot and E as the summit. Let A, B, C represent the three mile markers and let P represent the foot of the perpendicular from the foot of the mountain D to the road R. Triangles ADE, BDE, CDE are right triangles with angle D being a right angle. Also triangles PAD, PBD, PCD are right angles with angle P being a right angle. All distances are being measured in miles. Using trigonometry and the labels as shown above, we have,

$$
\begin{aligned}
x &= h \cot \gamma. \\
y &= h \cot \beta. \\
z &= h \cot \alpha. \\
x^2 &= p^2 + q^2. \\
y^2 &= p^2 + (q+1)^2. \\
z^2 &= p^2 + (q+2)^2. \\
\Rightarrow x^2 - 2y^2 + z^2 &= 2. \\
\Rightarrow h^2 \left[\cot^2 \alpha - 2\cot^2 \beta + \cot^2 \gamma\right] &= 2. \\
\Rightarrow h &= \left[\frac{2}{\cot^2 \alpha - 2\cot^2 \beta + \cot^2 \gamma}\right]^{1/2}.
\end{aligned}
$$