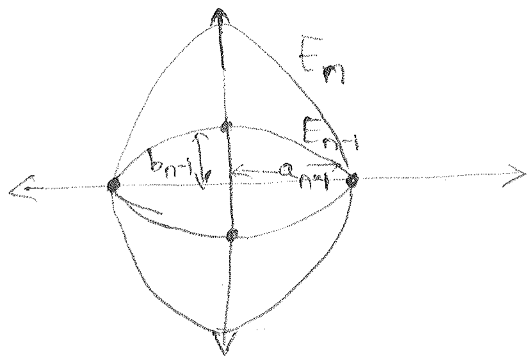
Solution to Problem of the Month, March 2009



Let a_n, b_n denote the lengths of the semi-axes of the n^{th} ellipse.

Then,

$$b_n^2 = a_{n-1}^2 \quad ; \quad a_n e_n = b_{n-1} \qquad — (1)$$

Also,

$$b_n^2 = a_n^2 (1 - e_n^2) \quad ; \quad b_{n-1}^2 = a_{n-1}^2 (1 - e_{n-1}^2) \qquad — (2)$$

$$\Rightarrow \quad a_{n-1}^2 = a_n^2 (1 - e_n^2) \quad , \quad a_n^2 e_n^2 = a_{n-1}^2 (1 - e_{n-1}^2)$$

$$\Rightarrow \quad 1 - e_n^2 = \frac{e_n^2}{1 - e_{n-1}^2} \qquad — (3)$$

If $\{e_n\}$ converges to a limit, e^*, as $n \to \infty$,

then, $(1 - e^{*2})^2 = e^{*2} \quad \Rightarrow \quad e^{*2} + e^* - 1 = 0.$

$$\Rightarrow \quad e^* = \frac{1}{\tau}$$

Now, $e_n^2 = 1 - \dfrac{1}{2 - e_{n-1}^2}$ {from (3)} —(4)

Similarly, $e_{n-1}^2 = 1 - \dfrac{1}{2 - e_{n-2}^2}$ —(5)

So, $$e_n^2 - e_{n-1}^2 = \frac{-\left(e_{n-1}^2 - e_{n-2}^2\right)}{\left(2 - e_{n-1}^2\right)\left(2 - e_{n-2}^2\right)}$$

Thus, $e_n^2 - e_{n-1}^2$ and $e_{n-1}^2 - e_{n-2}^2$ have opposite signs.

So,

$$e_2 > e_1 \Rightarrow e_3 < e_2,\ e_4 > e_3,\ e_5 < e_4, \ldots$$

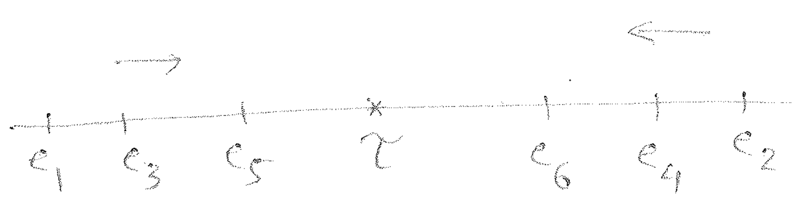
and

$$e_2 < e_1 \Rightarrow e_3 > e_2,\ e_4 < e_3,\ e_5 > e_4, \ldots$$

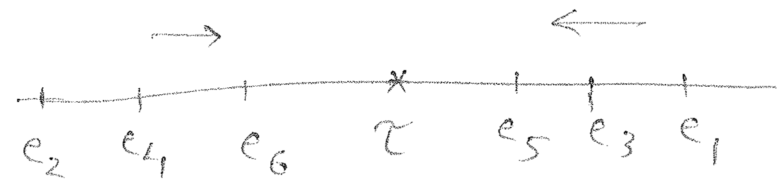
Also, since $e_{n-1}^2 < 1$, $e_{n-2}^2 < 1$, $2 - e_{n-1}^2 > 1$, $2 - e_{n-2}^2 > 1$, we have

$$\left|e_n^2 - e_{n-1}^2\right| < \left|e_{n-1}^2 - e_{n-2}^2\right|$$

so that when $e_2 > e_1$, we get



and when $e_2 < e_1$, we get



Thus, in either case one of $\{e_1, e_3, e_5, \dots\}$ and $\{e_2, e_4, e_6, \dots\}$ is monotonic increasing and other is monotonic decreasing.

Also, the monotonic increasing sequence is bounded above and the monotonic decreasing sequence is bounded below so that both sequences converge to finite limits (say e^{**} and e^{***})

Then, we get,

$$e^{**2} = \frac{1 - e^{***2}}{2 - e^{***2}}, \qquad e^{***2} = \frac{1 - e^{**2}}{2 - e^{**2}}$$

(From (4) and (5))

$\Rightarrow$ $e^{**} = e^{***}$ and further that

(4) $e^{**^2}(2 - e^{**^2}) = 1 - e^{**^2}$

$$\Rightarrow \quad e^{**^4} - 3e^{**^2} + 1 = 0$$

$$\Rightarrow \quad e^{**^2} = \frac{3 - \sqrt{5}}{2}$$

$$\Rightarrow \quad e^{**} = \frac{\sqrt{5} - 1}{2} = \frac{1}{\tau} = e^{*}$$

$\blacksquare$