Integral Equations and the Simple Harmonic Oscillator

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Abstract

The energy eigenstates for the simple harmonic oscillator in quantum mechanics are found through the solution of an integral equation.

The simple harmonic oscillator (SHO) is one of a handful of systems that are both physically relevant and amenable to an exact treatment by relatively simple methods. Textbook discussions (see, e.g., Refs. 1-4) typically develop the energy eigenfunctions either by direct solution of the Schrödinger equation, or by the more abstract algebraic method. The path integral approach may be found in more advanced books,[5] or in the pages of this Journal[6]. The purpose of this note is to introduce a procedure based on integral equations whose simplicity and elegance is comparable to the algebraic method. Aside from being of interest in its own right, this procedure may provide students with a useful introduction to mathematical techniques relevant to more difficult subjects like, e.g., the formal theory of scattering.

Let us consider the Schrödinger equation for the energy eigenstates of the one-dimensional SHO:

$$-\frac{\hbar^2}{2m}\,\frac{d^2\psi}{dx^2} \;+\; \tfrac{1}{2}\,m\omega^2 x^2\,\psi \;=\; E\,\psi \tag{1}$$

We recall that ψ may simultaneously be chosen to be a parity eigenstate, $\psi(-x) = \pm\psi(x)$.

Changing to the dimensionless variable $y = (m\omega/\hbar)^{1/2}\,x$ leads to the simpler form

$$\psi'' \;+\; (\gamma^2 \;-\; y^2\,)\,\psi \;=\; 0 \tag{2}$$

where primes indicate differentiation with respect to y, and $\gamma = (2E/\hbar\omega)^{1/2}$.

Taking the Fourier transform of Eq.(2) we find, with

$$g(k) \;=\; \frac{1}{\sqrt{2\pi}}\,\int_{-\infty}^{\infty}\,\psi(y)\,e^{iky}\,dy \tag{3}$$

and the assumption that ψ and ψ' vanish as $|y| \to \infty$:

$$-k^2\,g(k) \;+\; \gamma^2\,g(k) \;+\; \frac{d^2 g(k)}{dk^2} \;=\; 0 \tag{4}$$

Eq.(4) is formally identical to Eq.(2). Furthermore, since quantities like the total probability and the expectation value of the potential energy must remain finite for finite E, we should demand $g(k),\ dg(k)/dk \to 0$ as $|k| \to \infty$. It follows that g and ψ differ at most by a normalization constant,

$$g(k) \;=\; c\,\psi(k) \tag{5}$$

i.e., ψ satisfies the integral equation

$$c\,\psi(k) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(y)\, e^{iky}\, dy \tag{6}$$

The constant may now be determined by substituting $c\,\psi(y)$ on the right hand side:

$$\begin{aligned} c^2\,\psi(k) \;&=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(z)\, e^{izy}\, e^{iky}\, dz\, dy \\ &=\; \int_{-\infty}^{\infty} \psi(z)\, \delta(z+k)\, dz \\ &=\; \psi(-k) \end{aligned} \tag{7}$$

Eigenstates of even parity are therefore associated with $c^2 = 1$, or $c = \pm 1$, whereas eigenstates of odd parity require $c^2 = -1$, or $c = \pm i$. We summarize these alternatives by writing $c = i^p$, $p = 0, 1, 2, 3$, so that states of even (odd) parity are given by even (odd) values of p. Note that the present approach has shifted the emphasis from ψ as an energy eigenstate, to ψ as a parity eigenstate.

There are several ways one can solve the integral equation (6); because of its close connection to the algebraic method, we have chosen an approach that relies on elementary properties of Fourier transforms. Let $\mathcal{F}\{\cdot\}$ denote the Fourier transform operator. Then

$$\begin{aligned} \mathcal{F}\{\psi'(y)\} \;&=\; -ik\, g(k) \\ \mathcal{F}\{y\,\psi(y)\} \;&=\; -i\, \frac{dg(k)}{dk} \end{aligned} \tag{8}$$

Adding and using the notation $L_\pm = u \pm d/du$, where $u = y$ or k:

$$\mathcal{F}\{L_\pm\,\psi(y)\} \;=\; -iL_\pm\, g(k) \tag{9}$$

In our case $g(k) \;=\; i^p\,\psi(k)$, so

$$\mathcal{F}\{L_\pm\,\psi(y)\} \;=\; i^{p+3}\, L_\pm\,\psi(k) \tag{10}$$

Equation (10) tells us that if ψ is a solution of Eq. (6), then $L_\pm\,\psi$ are also solutions (with opposite parity, since p and $p' = p+3$ cannot both be even or odd). It should be obvious that this argument may be extended to show that $L_\pm^n\,\psi$, $n = 0, 1, 2, \ldots$, are also solutions to our problem. Thus all we need to find is one state satisfying Eq. (6); the rest may be generated by successive applications of L_+ or L_-. The well-known formula

$$e^{-k^2/4a} \;=\; \sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} e^{-ay^2-iky}\,dy \tag{11}$$

suggests the use of $a = 1/2$ to provide such a "seed" state. Substitution of the ansatz $\psi_0(y) = A_0\,e^{-y^2/2}$ into Eq. (6) confirms this, provided $p = 0$. The constant A_0 is fixed, as usual, by normalization with respect to x, $A_0 = (m\omega/\pi\hbar)^{1/4}$. The next eigenstate is

$$\begin{aligned}\psi_1(y) &= A_1\,L_-\,e^{-y^2/2}\\ &= -A_1\left(e^{y^2/2}\,\frac{d}{dy}\,e^{-y^2/2}\right)e^{-y^2/2}\\ &= 2\,A_1\,y\,e^{-y^2/2}\end{aligned} \tag{12}$$

Normalization together with a conventional choice for the phase yield the odd-parity, $p = 3$ eigenstate $\psi_1(y) = (m\omega/4\pi\hbar)^{1/4}\;y\,e^{-y^2/2}$. Continued applications of L_- lead to the standard solutions for the SHO eigenstates, i.e., the Hermite orthogonal functions

$$\psi_n(y) \;=\; \left(\frac{m\omega}{4^n\,(n!)^2\,\pi\hbar}\right)^{1/4} L_-^n\;e^{-y^2/2} \;=\; \left(\frac{m\omega}{4^n\,(n!)^2\,\pi\hbar}\right)^{1/4} H_n(y)\;e^{-y^2/2} \tag{13}$$

with H_n the Hermite polynomials. No new solutions are generated by application of L_+, since $L_+\,\psi_0 \;=\; 0$. An alternative to the above is to begin with the trivial solution to Eq. (6), $\psi_{-1} = 0$ (the subindex is chosen so as to match the conventional notation for ψ_n), and use the operator L_+. Integration of $\psi_{-1} = L_+^n\;\psi_n$ leads back to Eq. (13) if we impose the condition of orthonormality on the eigenstates.

The alert reader may be somewhat puzzled by the fact that while we have often mentioned parity in our discussion, the description by means of the integral equation (6) involves *four* distinct eigenvalues i^p. These eigenvalues are conserved by the evolution of the system since the Hamiltonian commutes with the Fourier operator (compare Eqs. (2) and (4)): the integral equation approach has thus automatically uncovered a deeper symmetry of the SHO. This should not, however, be construed as proving the superiority of the integral equation point of view. Indeed, in the case of the SHO it is much easier to find the full symmetry group if we write the Hamiltonian in terms of dimensionless momentum and coordinate operators p and q, $H = \hbar\omega\,(p^2 + q^2)/2$, or in terms of creation and destruction operators $a^\dagger$ and a, $H = \hbar\omega\,(a^\dagger a + \frac{1}{2})$. The first form is clearly invariant under proper rotations in the p, q space (improper rotations cannot be allowed because they do not preserve the commutation relation $[q, p] = i$), so the full (dynamical) symmetry group is $SO(2)$. The second form is preserved by the unitary transformation $a \rightarrow e^{i\alpha}\, a$, $a^\dagger \rightarrow a^\dagger\, e^{-i\alpha}$, implying a $U(1)$ symmetry.[7] The latter form generalizes easily to the N dimensional isotropic case, leading to the symmetry group $U(N)$.[8] Our integral equation method has therefore uncovered only that subgroup - usually labeled C_4 - of the complete symmetry group which corresponds to the four transformations $q \rightarrow q$, $p \rightarrow p$; $q \rightarrow -q$, $p \rightarrow -p$; $q \rightarrow -p$, $p \rightarrow q$; and $q \rightarrow p$, $p \rightarrow -q$. On the other hand, there are instances where an integral equation formulation does provide a considerable advantage over more straightforward methods. A case in point is the analysis of the hydrogen atom by Fock[9] showing that the symmetry group is the four-dimensional rotation group $SO(4)$. The interested reader may wish to consult Refs. 8 and 10-12 on this point.

[1]L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968).

[2]E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970).

[3]J. J. Sakurai, *Modern Quantum Mechanics*, Revised Edition (Addison-Wesley, Reading, Massachusetts, 1994).

[4]L. E. Ballentine, *Quantum Mechanics* (Prentice Hall, Englewood Cliffs, New Jersey, 1990).

[5]B. R. Holstein, *Topics in Advanced Quantum Mechanics* (Addison-Wesley, Redwood City, CA, 1992).

[6]J. F. Donoghue and B. R. Holstein, "The harmonic oscillator via functional techniques,"

Am. J. Phys. **56**, 216-222 (1988).

[7]The equivalence of these two results is guaranteed by the isomorphism between $SO(2)$ and $U(1)$.

[8]M. I. Petrashen and E. D. Trifonov, *Applications of Group Theory in Quantum Mechanics* (M.I.T. Press, Cambridge, Massachusetts, 1969).

[9]V. Fock, "Zur Theorie des Wasserstoffatoms," Z. Physik. **98**, 145-154 (1935).

[10]M. J. Englefield, *Group Theory and the Coulomb Problem* (John Wiley & Sons, New York, 1972).

[11]J. F. Cornwell, *Group Theory in Physics*, Vol. II (Academic Press, London, 1984).

[12]R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (John Wiley & Sons, New York, 1974).